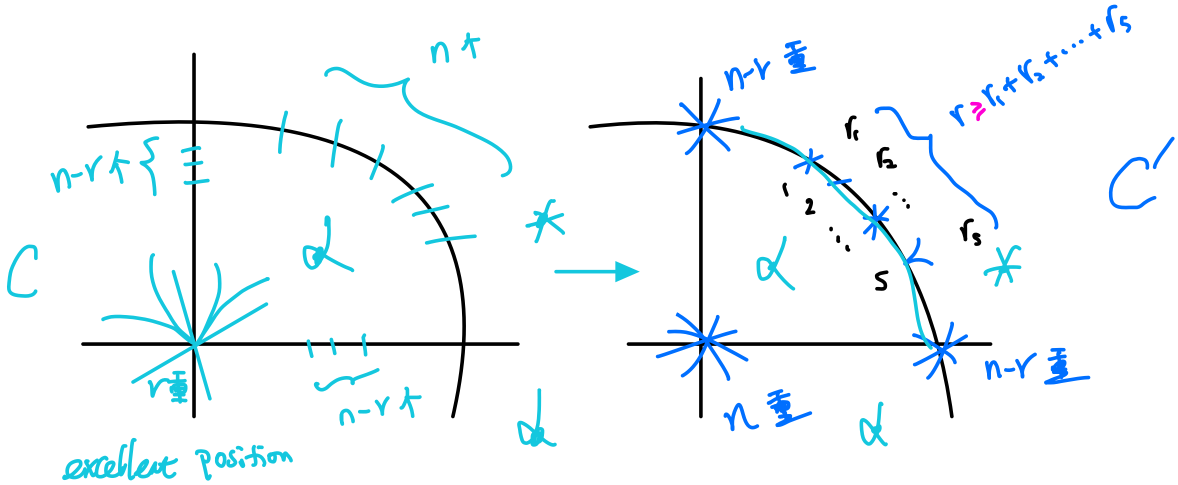


Fact: Denote $g^*(C) := \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P-1)}{2}$. Then

$$\boxed{g^*(C') := g^*(C) - \sum_{i=1}^{s} \frac{r_i(r_i-1)}{2}}$$

$r_i \neq 1$ $\Downarrow$ $\boxed{g^*(C') < g^*(C)}$

pf: $\deg(C') = 2n-r$

Singular pts on C' :
- $[0:0:1]$ & $[0:1:0]$ & $[1:0:0]$
- $Z=0$ & $X \neq 0, Y=0$.
- the one coming from $C \setminus [0:0:1]$

$$g^*(C') = \frac{(2n-r-1)(2n-r-2)}{2} - \frac{n(n-1)}{2} - 2 \cdot \frac{(n-r)(n-r-1)}{2} - \sum_{P \in C \setminus [0:0:1]} \frac{r_P(r_P-1)}{2}$$

$(g, \#\{\text{non ordinary pts}\}) \downarrow$

Thm 3 C = proj curve. Then $\exists!$ nonsingular proj. curve X & birational morphism $f : X \twoheadrightarrow C$.

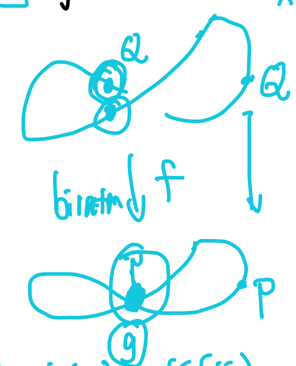
$\leftarrow$ $\boxed{\text{nonsingular model}}$ of C

DVR

$\downarrow$

$\mathcal{O}_Q(X)$

X



Notation: $f : X \to C$ $\leftarrow$ plane curve. $f(Q) = P \in C$

$$\mathrm{ord}_Q(G) := \mathrm{ord}_Q(g)$$

$C \hookrightarrow \mathbb{P}^2 \hookleftarrow G$

$g \in K(C) = K(X)$

$G_* \bmod I(C) = g \in K(C)$

Def: 1) pts of X will be called places of C (or of K)

2). A place Q is centered at P if $f(Q) = P$

$\mathrm{Spec}\, \mathcal{O}_K$

K

$$= \mathcal{O}_P(\mathbb{A}^2)/(F_*, G_*) \cong \left(\mathcal{O}_P(\mathbb{P}^2)/F_*\right)/\left((F_*, G_*)/F_*\right)$$

$$= \mathcal{O}_P(F)/(g)$$

$P \in F$ simple $\Rightarrow I(P, F \cap G) = \mathrm{ord}_P^F(G)$

what about when $P \in F$ is not simple?

Prop 2. C = Irr. proj. plane curve, $P \in C$, $f : X \to C$ nonsingular model, G = plane curve (possibly reducible). Then

$$I(P, C \cap G) = \sum_{Q \in f^{-1}(P)} \mathrm{ord}_Q(G)$$

$$\begin{array}{ccc} \{Q_1, \ldots, Q_s\} & X \supset & f^{-1}(P) \\ \downarrow & \downarrow f & \\ P & C \ni \{P\} & \end{array}$$

§8. Riemann-Roch Theorem

Setup: C = irr. proj. plane curve with nonsingular model $f: X \to C$. $K = k(C) = k(X)$.

$P \in X$, ord_P = order function on K.

§8.1. divisors

$F \cdot G = \sum_P I(P, F\cap G) \cdot P$

if F = proj ~~smooth~~

$F \cdot G$ divisor on F

(Weil) divisor on X

$$D = \sum_{P \in X} n_P \cdot P$$

$n_P \in \mathbb{Z}$ & $n_P = 0$ for almost all $P \in X$.

$$\deg(D) := \sum_{P \in X} n_P \in \mathbb{Z}$$

Fact: $\text{Div}(X)$ the set of all divisors on X forms an abelian gp.

free abelian gp on the set X.

$$\sum n_P \cdot P \geq \sum m_P \cdot P \overset{\text{def}}{\iff} n_P \geq m_P \quad \forall P.$$

$$\underset{\sum n_P \cdot P}{\overset{\|}{D}} = \text{effective (or, positive)} \overset{\text{def}}{\iff} D \geq 0. \ (\text{i.e. } n_P \geq 0, \forall P)$$

Example: C = plane curve of deg n.

G = plane curve not containing C as a component. (of deg m)

$$\text{div}(G) := \sum_{P \in X} \underbrace{\text{ord}_P(G)}_{\geq 0} \cdot P \in \text{Div}(X) \qquad \sum_{P' \in C} I(P', C \cap G) = mn$$

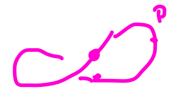
Fact: $\text{div}(G)$ is of deg. mn. (m: deg C, n: deg G.)

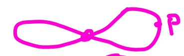
pf: $\deg(\text{div}(G)) = \sum_{P \in X} \text{ord}_P(G) = \sum_{P' \in C} \left(\sum_{P \in f^{-1}(P')} \text{ord}_P(G) \right)$, where $\sum_{P \in f^{-1}(P')} \text{ord}_P(G) = I(P', C \cap G)$

Example: (principal divisors) $\forall z \in K^*$, divisor of z is defined as

$$\boxed{div(z) = \sum_{P \in X} ord_P(z) \cdot P}$$

$\boxed{ord_P(z) = 0}$





$P(X) := \{ div(z) \mid z \neq 0 \} \subset Div(X)$ subgroup

$\boxed{ord_P(z) = 0 \text{ for almost simple pt } P \in C.}$

if not.

$\exists$ only pt $P_1, \cdots$ s.t. $z(P_i) = 0$

Basic Facts:

1) $div(z z') = div(z) + div(z')$
2) $div(z^{-1}) = -div(z)$.
3) $deg(div(z)) = 0$ $\quad z \in K = K(C)$

$$z = \frac{F \bmod I(C)}{G \bmod I(C)} \quad deg F = deg G.$$

$\Downarrow$

$div(z) = div(F) - div(G)$

Def: Two divisor D, D' are called *linear equivalent* if

$$D' = D + div(z)$$

for some $z \in K^*$. In which case, write $D' \equiv D$.

Weil divisor class group $\quad Cl(X) = Div(X)/\equiv \;=\; Div(X)/P(X)$

$$\boxed{0 \to k^* \to K^* \xrightarrow{div} Div(X) \to Cl(X) \to 0}$$

$$div(z) = 0 \overset{z \in K^*}{\Leftrightarrow} z \in k^*$$

$\Leftarrow$) ✓

$\Rightarrow$) :

$div(z) = div_0(z) - div_\infty(z)$

z

$z = \frac{F}{G}$

$$div(z) := \sum_{P \in X} ord_P(z)\, P$$

$$div_0(z) := \sum_{\substack{P \in X \\ ord_P(z) \geq 0}} ord_P(z) \cdot P \geq 0$$

$$div_\infty(z) := \sum_{\substack{P \in X \\ ord_P(z) < 0}} (-ord_P(z))\, P \geq 0$$

$D = \sum_P n_P \cdot P \in \mathrm{Div}(X)$.

$\boxed{\begin{array}{l} 1^\circ \ n_P > 0 \quad \mathrm{ord}_P(f) \geq -n_P \Rightarrow f \\ 2^\circ \ n_P < 0 \quad \mathrm{ord}_P(f) \geq \underbrace{-n_P}_{>0} \Rightarrow f \end{array}}$

$L(D) := \{ f \in K^* \mid \mathrm{ord}_P(f) \geq -n_P \text{ for all } P \in X \} \cup \{0\}$.

↳ set of rational functions with poles only at the chosen points and with poles no worse than order n_P at P.

Fact: 1) $f \in L(D) \Leftrightarrow f = 0$ or $\mathrm{div}(f) + D \geq 0$

2) $L(D)$ forms a v.s. over k.

$\left\{ \begin{array}{l} f \in L(D) \Rightarrow \underline{\lambda f \in L(D)} \quad \checkmark \quad (\lambda \in K^*) \\ \underline{f, g \in L(D)} \Rightarrow \underline{f+g \in L(D)} \end{array} \right.$

$\underline{\mathrm{ord}_P(f+g) \geq \min(\mathrm{ord}_P(f), \mathrm{ord}_P(g))}$

$\ell(D) := \dim_k L(D)$

aim: calculate $\ell(D)$.

Prop 3. (1). $D \leq D' \Rightarrow L(D) \subset L(D')$ & $\boxed{\dim_k (L(D')/L(D)) \leq \deg(D'-D)}$

(2). $\underline{L(0) = k}$; $L(D) = 0$, if $\underline{\deg(D) < 0}$.

(3). $\deg(D) \geq 0 \Rightarrow \ell(D) \leq \deg(D) + 1$.

(4). $D \equiv D' \Rightarrow \ell(D) = \ell(D')$

Pf (1): $\forall f \in L(D) \Rightarrow \mathrm{div}(f) + D \geq 0 \Rightarrow \mathrm{div}(f) + D' \geq 0 \Rightarrow f \in L(D')$

Assume $D' = D + P$ $\boxed{D' = mP + \sum_{Q \neq P} n_Q \cdot Q}$ t, $\mathrm{ord}_P(t) = 1$

$\boxed{0 \to L(D) \to L(D') \xrightarrow{\varphi} k}$ $t^m z$

$z \longmapsto t^m z \,\big|_P$ $zt^m \in \mathcal{O}_P(X)$, $\mathrm{ord}_P(zt^m) \geq 0$

$\boxed{m_P(X) \to \mathcal{O}_P(X) \to k, \quad f \mapsto f(P)}$

$\Uparrow$

$\mathrm{div}(z) + D' \geq 0 \Rightarrow \mathrm{ord}_P(z) + m \geq 0 \Rightarrow \underline{\mathrm{ord}_P(z) \geq -m}$

② $\operatorname{div}(f) + D \geq 0$ $\quad f \in L(D) \setminus \{0\}$

$\Rightarrow \deg(\operatorname{div}(f)) + \deg(D) \geq 0$

$\overset{\shortparallel}{0}$

$\Rightarrow \operatorname{div}(D) \geq 0$

$\underbrace{0 \to P_1 \to P_1 + P_2 \to \cdots\cdots}_{\deg D} \to D = P_1 + \cdots + P_r$

$L(0) \to L(P_1) \to \cdots$

$+1 \quad +1$

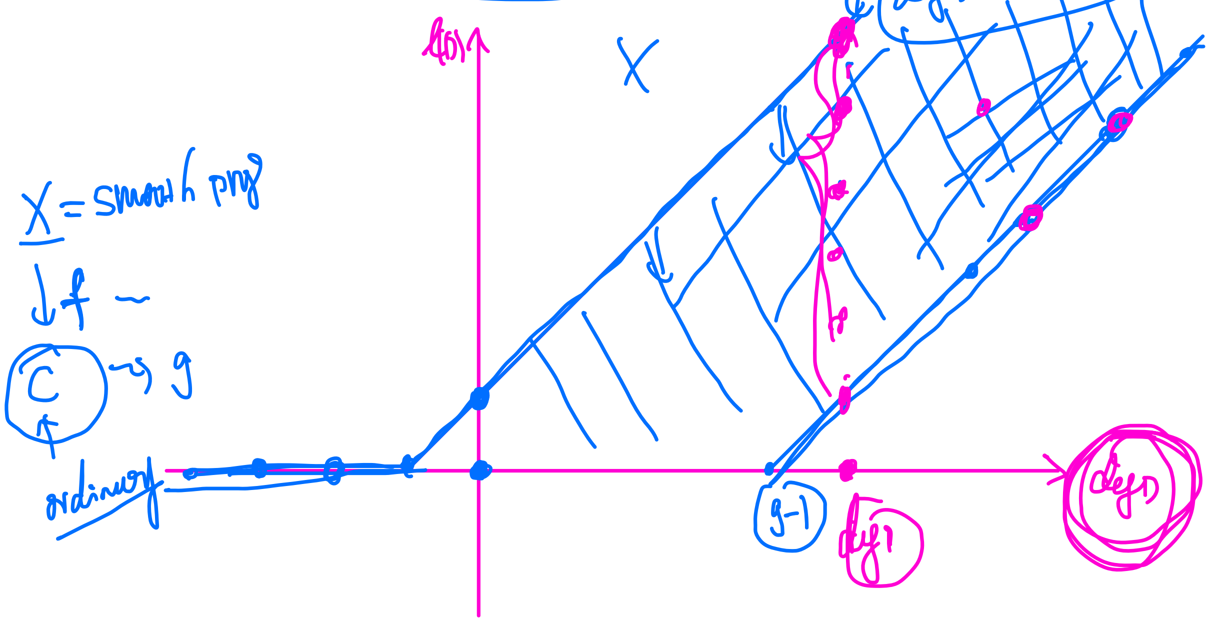
$+0 \quad +0$

$\underline{L(D)} \leq \deg D + \deg L(0)$

$\qquad\qquad \shortparallel$

$\qquad\qquad \deg D + 1$

$D \rightsquigarrow \underline{(\deg D, \ell(D))}$



Thm (Riemann's thm) $\exists\, g$ integer s.t. $\ell(D) \geq \deg(D) + 1 - g$ for all D.

$$\boxed{g = \max_{D} \{ \deg D + 1 - \ell(D) \} \in \{0, 1, 2, \cdots\}}.$$

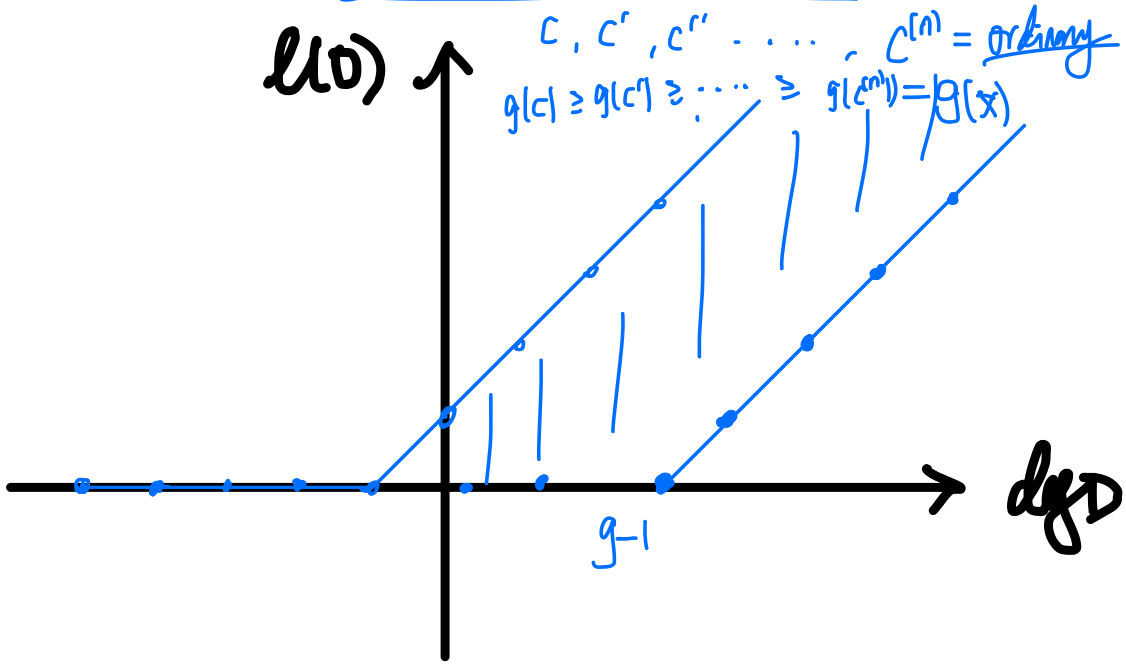
is called the genus of X (or of K, or of C)

Prop C = plane curve with only ordinary multiple pts. n = deg. of C, $r_P = m_P(C)$. Then

$$\boxed{g = \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P - 1)}{2}}.$$

Cor 1: C = plane curve of deg n. $r_P = m_P(C)$, $P \in C$. Then

$$g \leq \boxed{\frac{(n-1)(n-2)}{2} - \sum \frac{r_P(r_P-1)}{2}} \qquad g^*(C)$$



§8.4. Derivation and differentials

algebraic background to study differentials on a curve

$df = f_{x_1} dx_1 + f_{x_2} dx_2 + \cdots + f_{x_n} dx_n$

R = ring containing k.

$\sum_{x \in R} r_x [x]$ $\quad r_x \in R$

R-mod

$\Omega_k(R) := F / N$

N: submodule of F generated by ① ② ③

F: free R-module on set $\{[x] \mid x \in R\}$

$\Omega_k(R)$: module of differentials

①: $[x+y] - [x] - [y]$

②: $[\lambda x] - \lambda [x]$

③: $[xy] - x[y] - y[x]$.

$$d : R \to \Omega_k(R)$$
$$x \mapsto \overline{[x]} =: dx$$

$[x] \in F \twoheadrightarrow F/N$

$[x] \mapsto dx$

- $d(x+y) = \overline{[x+y]} = \overline{[x]} + \overline{[y]} = dx + dy$
- $d(\lambda x) = \lambda dx$

$d(xy) = y dx + x dy$

Fact: 1) $R = k[x_1, \cdots, x_n] \Rightarrow \Omega_k(R) = \sum_{i=1}^{n} R \cdot dx_i$

2) $K = k(x_1, \cdots, x_n) \Rightarrow \Omega_k(K) = \sum_{i=1}^{n} K\, dx_i$

$f(x_1, \cdots, x_n)$

$df = \sum f_{x_i} dx_i$

$K = k(x, y) \Rightarrow \Omega = k\,dx + k\,dy$

Prop 6 1). K = alg. function field in one variable over k. Then

$\Omega_k(K)$ = 1-dim. vect. sp. over K

2). (char $k = 0$). $x \in K \setminus k \Rightarrow \Omega_k(K) = K \cdot dx$

$\Rightarrow$ one can define $\frac{df}{dt}$

$f(y)=0$ $\qquad$ $\underline{F(x,y)=0}$ $\qquad$ $dx \neq 0$

$\Downarrow$

$dF(x,y)=0$

$\Rightarrow \underline{F_x(x,y)\,dx + F_y(x,y)\,dy = 0}$

$$\boxed{\text{char } k = p \qquad \underline{x^p \in K|k} \qquad \Rightarrow dx^p = p x^{p-1} = 0}$$

$\deg = 0$

principal divisor

$\forall\ z \in K^* \rightsquigarrow \operatorname{div}(z)$

$\deg = 2g-2$

$w \in \Omega_k(K) \rightsquigarrow \operatorname{div}(w)$

canonical divisor

$\sum n_p P$

§8.5 Canonical Divisors.

$w = f \cdot dt$ $\exists! f \in K$

X = nonsingular model of a projective curve C, with function field K

$w \in \Omega = \Omega_k(K)$. ($w \in \Omega$ is called differential on X (or on C))

$w \neq 0$

$\mathrm{ord}_P(w) := \boxed{\mathrm{ord}_P(f)}$ $\boxed{w \in \Omega_K(K) = K \cdot dt}$

if $w = f\,dt$ for some uniformizer t in $\mathcal{O}_P(X)$. $=$ DVR $dt \neq 0$

well-defined: $u \sim t \Rightarrow f\,dt = w = g\,du \Rightarrow f/g = \frac{du}{dt} \in \mathcal{O}^\times \Rightarrow v$.

$$\mathrm{div}(w) := \sum_{P \in X} \mathrm{ord}_P(w) \cdot P \in \mathrm{Div}(X) \quad (w \neq 0)$$

well-defined (prop 8). a canonical divisor

Fact: W = a canonical divisor. Then

$$\boxed{\deg(W) = 2g - 2}$$

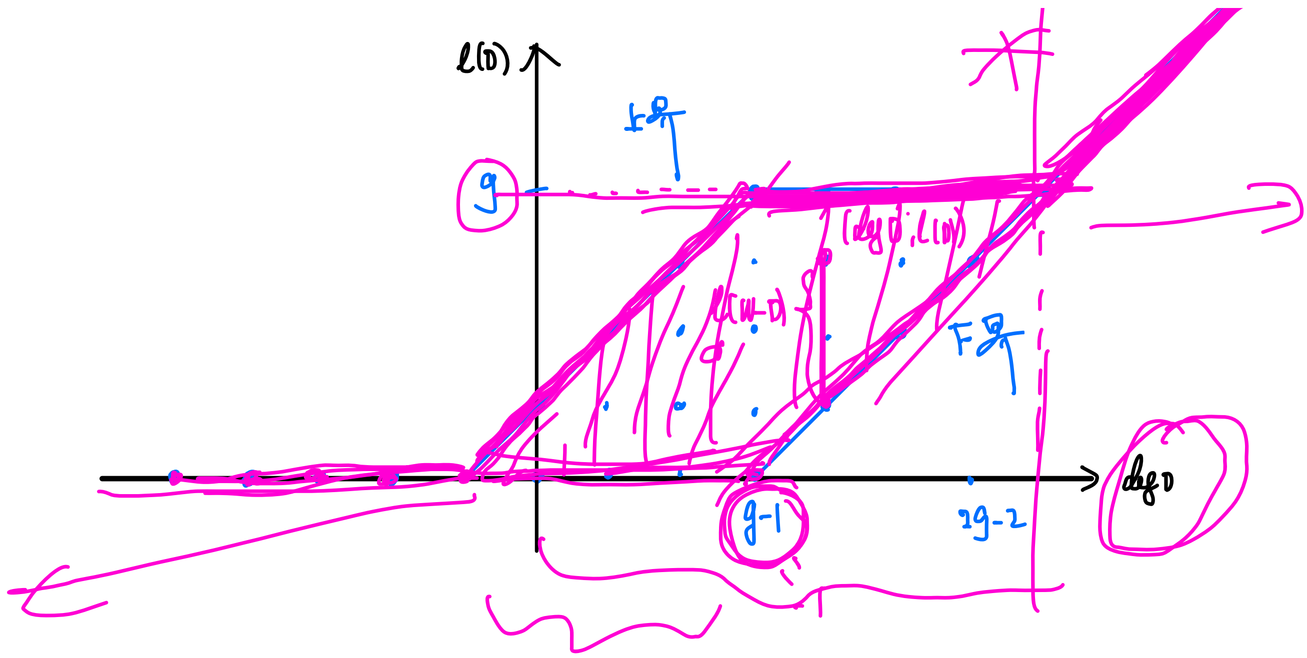
find missing term in Riemann's thm.

Thm (Riemann-Roch thm) W = canonical divisor on X. Then

$$\ell(D) = \deg D + 1 - g + \ell(W - D)$$

Rmk: • holds for $D \gg 0$ or $D \ll 0$.

• compare both sides for D & $D+P$.



1° $\deg D \in [0, \cdots, g-1]$ ✓

2° $\deg D \in [g, \cdots, 2g-2]$?

3° $\deg D \geq 2g-1 \Rightarrow \ell(D) = \deg D + 1 - g$

3° $\ell(D) = \deg D + 1 - g + \ell(W-D)$ $=0$

$\deg(W-D) = \deg W - \deg D$

$\leq (2g-2) - (2g-1) = -1$

2° $\ell(D) = \deg D + 1 - g + \ell(W-D)$

$\deg(D) + \deg(W-D) = \deg W = 2g-2$

$\deg D \in [g, \cdots, 2g-2]$

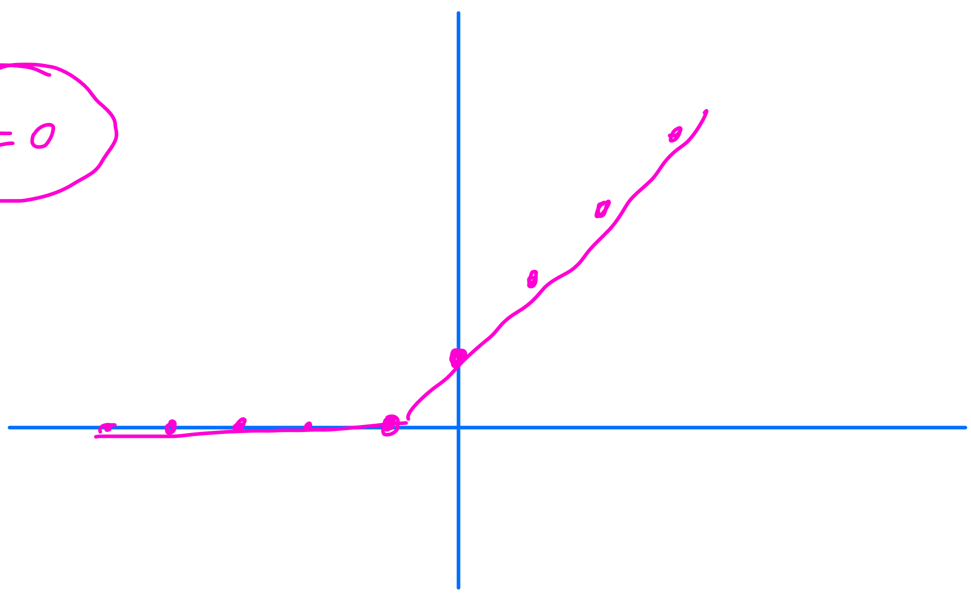
$\Rightarrow \deg(W-D) = [0, 1, \cdots, g-2]$

$\Rightarrow \ell(D) \leq \deg(W-D) + 1$

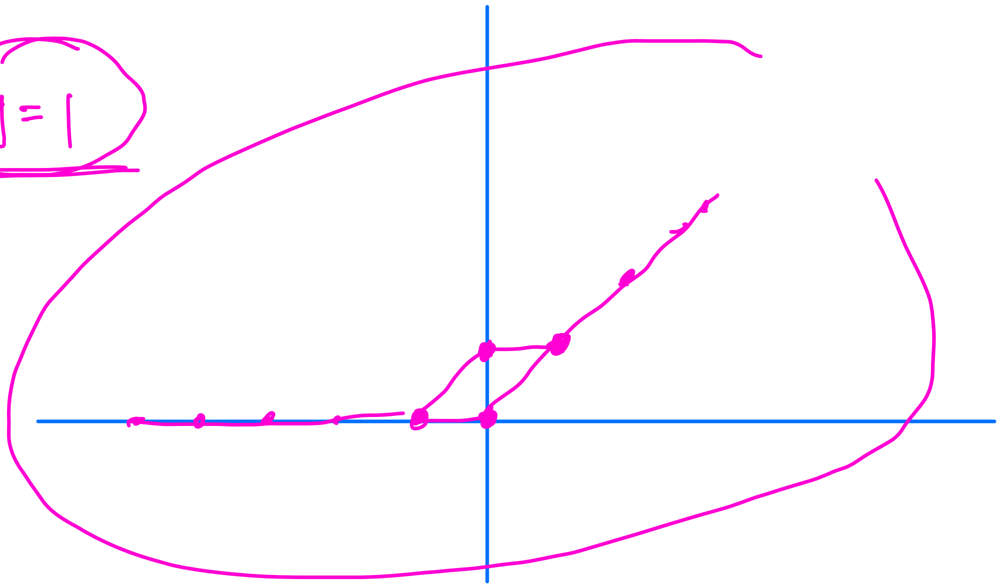
$\ell(D) \leq \deg D + 1 - g + \deg(W-D) + 1$

$= 2g-2+2-g = g$

$g=0$



$g = 1$



$\ell(D) \to \deg D$

$\deg D = 0$

$g = 2$